Study on the Numerical Solution and Application of Fractional Partial Differential Equations

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Abstract: This paper mainly studies the numerical calculation methods of several types of time fractional partial differential equations and some applications of fractional calculus theory and numerical methods in science. First, for two-dimensional nonlinear fractional-order reaction sub-diffusion equations, we propose two compact finite difference schemes, and use Fourier analysis to give a theoretical analysis of the stability and convergence of these two schemes. Secondly, for the first-order fractional Stokes problem of Cantonian second-order fluid under heating, we propose a numerical parameter estimation method to estimate the order of the Riemann-Liouville fractional derivative. Thirdly, for the two-dimensional fractional-order Cable equation, we propose a compact fourth-order finite-difference scheme in space, and use Fourier analysis to give a theoretical proof of stability and convergence. For the inverse problem, we propose a numerical parameter estimation method, and give the optimization of two fractional order derivatives. In the tumor hyperthermia experiment, we constructed a time-fractional heat wave model of the double-layered spherical tissue, and used the implicit difference method to give a numerical solution of the T model. For the inverse problem, with the help of thermal experimental data, we propose a nonlinear parameter estimation method that gives an optimal estimate of the unknown fractional derivative and relaxation time parameters. Finally, the transport process of steel ions across the intestinal wall, we established a spatial fractional order anomalous diffusion model under the action of concentration gradient and potential gradient, and obtained the numerical solution of the problem by finite difference method.

1. Introduction

With the emergence of fractional calculus in more and more subject areas, the solution and calculation of fractional order models become increasingly important. Analytical solutions of fractional partial differential equations can generally be obtained by integral transformation and inverse transformation methods, such as Laplace transformation, Fourier transformation, Mellin transformation, etc., but the solutions generally contain special functions, such as Mittag-Leffler function, H function, Wright functions, etc., it is very difficult to directly calculate these special functions in engineering, and for most fractional partial differential equations, it is generally difficult to find their analytical solutions. Therefore, it is extremely important to study the numerical solution of fractional partial differential equations. At present, the numerical methods for solving fractional partial differential equations mainly include finite difference method, finite element method, finite volume method, spectral method, meshless method, etc. The content of this paper mainly involves several types of time fractional partial differential equation models. The numerical solution is mainly calculated by finite difference methods. Therefore, the following article mainly describes several finite difference methods of time fractional partial differential equations. As for the finite element method, finite volume method, spectral method, meshless method and other methods for solving fractional partial differential equations.

2. Several Types of Fractional Derivative Models

Traditional diffusion equations associate the first derivative with respect to time and the second derivative with respect to space. Fractional diffusion equations use their fractional order similarity to replace the space and time derivatives. This section briefly introduces Three types of fractional

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diffusion models. It is worth noting that they are only a very small part of many existing models. The main feature of anomalous diffusion is its historical dependence and or global correlation and fractional calculus can describe this memory and or non-locality, so it has been widely used in the modeling of anomalous diffusion. In the development of fractional calculus, scholars based on different perspectives We give the definitions of different fractional derivatives, and now the most widely used are the definitions of Remiann-Liouville, Caputo, Grunwald-Letnikov, and generator types. The Remiann-Liouville and Caputo derivatives are defined in integral-differential form, of which the former It is mostly used to represent the fractional derivative of space, which plays a major role in the study of mathematical theory, and the latter is mostly used to represent the derivative of time fraction. Has the advantage of easy to add initial conditions; Grunwald-Letnikov derivatives are defined in the form of difference quotients, which are mainly used for numerical calculations; since the derivative of the subtype is usually closely related to the Levy process, it is defined in the form of integrals and is often used Yu represents the fractional derivative of space. It should be noted that these definitions can be transformed into each other under certain conditions.

A class of fractional diffusion equations that are currently widely used can be written as:

$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left(\kappa\, {}_xD_\theta^\alpha P(x,t)\right), \quad t>0,$$

$${}_0^CD_t^\gamma P(x,t) = \kappa\, {}_xD_\theta^\alpha P(x,t), \quad t>0.$$

Here $\gamma \in (0,1)$, $\alpha \in (0,2)$ and $|\theta| \leqslant \min\{\alpha, 2-\alpha\}$, $0D1-\gamma t$ and $C0D\gamma t$ denote respectively 1 $-\gamma$ order Remian-Liouville fractional derivative and γ order Caputo fractional derivative, and $xD\alpha\theta$ is often called fractional Riesz-Feller derivative, and its Fourier transform is:

$$\mathscr{F}[{}_xD_\theta^\alpha P(x,t)](\xi,t) = -|\xi|^\alpha e^{-i(\operatorname{sgn}\xi)\theta\pi/2}\mathscr{F}[P(x,t)](\xi,t),$$

When $\gamma = 1$ and $\alpha \in (0,2)$, (1.2) is a strict spatial fractional order equation, which includes two very important cases. One is when $\theta = 0$, for all $\alpha \in (0, 2)$ We have:

$$\frac{\partial P(x,t)}{\partial t} = -\kappa(-\Delta)^{\alpha/2}P(x,t),$$

$$(-\Delta)^{\alpha/2}P(x,t) := c_\alpha \int_{\mathbb{R}} \frac{P(x,t)-P(y,t)}{|x-y|^{1+\alpha}}dy$$

Called the α / 2-order fractional Laplace operator, it should be understood as the Cauchy principal value integral. Because (1.5) is a generator of the L´evy process with the characteristic function $e-\kappa t \mid \xi \mid \alpha$ in probability statistics, We also call it the derivative of the generator form. Here $c\alpha$ is a constant; the second is that when $\alpha =/1$, for all θ we have:

$$\frac{\partial P(x,t)}{\partial t} = \kappa\left(c_+\, {}_{-\infty}D_x^\alpha + c_-\, {}_xD_\infty^\alpha\right)P(x,t),$$

Where the coefficients $c\pm = \sin[(\alpha\mp\theta)\pi/2] \sin \alpha\pi$, and the operators $-\infty D\alpha x$ and $xD\alpha\infty$ denote the left and right Remiann-Liouville fractional derivative of order α, respectively, defined:

$${}_{-\infty}D_x^\alpha P(x,t) := \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^x \frac{P(\xi,t)}{(x-\xi)^{\alpha-n+1}}d\xi,$$

$${}_xD_\infty^\alpha P(x,t) := \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^\infty \frac{P(\xi,t)}{(\xi-x)^{\alpha-n+1}}d\xi.$$

Here n is the smallest integer greater than . The basic solutions of the Cauchy problem of equations (1.4) and (1.6) both statistically represent the probability that a particle doing the α-stable L´evy motion is at position t at time The mean square displacement of the motion is divergent, and the corresponding diffusion processes are called super-diffusion.

Of course, when $\gamma \in (0,1)$ and $\alpha \in (0,2)$, we can also obtain a spatio-temporal fractional model, which describes the tailing phenomenon and the spatial direction in the time direction that coexist in the diffusion process. The non-local correlation characteristics of the above, the type of anomalous diffusion at this time will depend on the magnitude relationship between $2\gamma\alpha$ and 1. Specifically, when $2\gamma\alpha$> 1, it is super-diffusion, and when $2\gamma\alpha$ <1, it is under-diffusion.

3. Numerical Solution of Fractional Partial Differential Equations

We consider the following class of fractional partial differential equations:

$$\frac{\partial^v u(x,t)}{\partial x^v} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = g(x,t),$$

Initial conditions: u (0, t) = u (x, 0) = 0,

Where 0 ⩽ x, t ⩽ 1, n \ 1 <v, γ ⩽ n, ∂vu (x, t) ∂xv and ∂γ u (x, t) ∂tγ are Caputo fractional differentials, respectively. g (x, t) is a known function, and u (x, t) is an evaluation function.

$$\frac{\partial^v u(x,t)}{\partial x^v} \approx \frac{\partial^v(\Phi^T(x)\mathbf{U}\Phi(t))}{\partial x^v} = \frac{\partial^v(\Phi^T(x))}{\partial x^v}\mathbf{U}\Phi(t)$$

$$= \left(\frac{\partial^v \Phi(x)}{\partial x^v}\right)^T \mathbf{U}\Phi(t) \approx \Phi^T(x)(\mathbf{D}^v)^T\mathbf{U}\Phi(t),$$

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} \approx \frac{\partial^\gamma(\Phi^T(x)\mathbf{U}\Phi(t))}{\partial t^\gamma} = \Phi^T(x)\mathbf{U}\frac{\partial^\gamma \Phi(t)}{\partial t^\gamma} \approx \Phi^T(x)\mathbf{U}\mathbf{D}^\gamma\Phi(t),$$

$$g(x,t) \approx \Phi^T(x)\mathbf{G}\Phi(t),$$

4. Applications of Fractional Partial Differential Equations

The establishment of fractional calculus theory has a history of more than 300 years, but it was not until the second half of the 20th century that it attracted the attention of technicians in the engineering field. At present, there is no unified definition form of fractional calculus. The more well-known definitions in the airspace. Forms: Riemamann-Liouville (R-L), Caputo, Grunwald-Letnikov (GL), and the definition form of Fourier transform. From the perspective of information theory, the physical meaning of fractional differential can be understood as generalized amplitude modulation and phase modulation, whose amplitude varies with the fractional power exponent with frequency, and the phase is the generalized Hilbert transform of frequency. In image processing, the integer-order differential operator is basically only suitable for processing high-frequency changes in the image, and does not have the ability to deal with discontinuous boundary points and have low-frequency variation characteristic details. However, the rich surface texture details in the image belong to the middle and low frequency components, and the integer order differential operator cannot handle them well. Studies have shown that when the differential fractional order v is in the range of 0 <v <1, the amplitude of the high-frequency component in the signal is sufficiently improved, the intermediate-frequency component is also strengthened, and the low-frequency and very low-frequency components are retained non-linearly. The smooth region in the image corresponds to the low frequency portion of the signal, the texture region in the image corresponds to the intermediate frequency portion of the signal, and the edge or noise region in the image corresponds to the high frequency portion of the signal. That is, using a fractional differential operator with an order of 0 <v <1 to process an image can not only better handle noise or edge information in the image, but also enhance detailed information such as texture in the image, and it can also retain Information about smooth areas in the image. In addition, after studying the mathematical model of the receptive field of human retinal ganglion cells in the literature, it was found that the receptive field model of fractional differential is more in line with the characteristics of human visual perception. The derivative is more refined and accurate. Therefore, the introduction of fractional calculus into the image reconstruction models of variational and partial differential equations can not only solve the problems of traditional methods, but also further apply it to the deep image segmentation, repair, and compression fields.

There are generally two methods to obtain partial differential equations: Gaussian smooth operator derivation and variational method derivation. The classical equations of partial differential equations derived by Gaussian smoothing operators are the anisotropic diffusion model (also called PM model) proposed by Perona and Malik; starting from the optimization problem, that is, the partial differential equations derived by the variational method The representative equation is the total variation (TV) regularization model proposed by Rudin et al. The current image processing models based on partial differential equations are mainly divided into two categories: one is a method of diffusion equation based on fluid diffusion theory; the other is a method of optimizing an energy functional based on variational method. The process of partial differential equations derived from Gaussian smoothing operators is relatively simple. The heat conduction equation derived by Gaussian filtering is a partial differential equation. The variational image reconstruction method needs to introduce an energy function to transform the image reconstruction problem into a functional to find the extremum problem. The main steps are: 1) Establishing the functional and its constraints from the physical problem; 2) Using functional variation, Find the Euler equation; 3) introduce the time variable, use the boundary conditions to establish a differential equation and solve it.

Although the PM model and the TV model can better retain the edge information of the image while denoising, the “step effect” will be generated in the processed image. It is precisely because of the characteristics of fractional calculus in image processing: it can handle high, medium and low frequency components in the image, so many scholars have introduced fractional calculus into the image processing model and got better results. Processing effect. Due to the ill-conditioned nature of the PM model's spread function, it is easy to produce a step effect in the processed image, which will cause the image quality to decrease. Therefore, scholars have made many improvements to the PM model. Some Literature Starting from reducing the “staircase effect” in the reconstructed image and maintaining the structural information in the image, using fractional calculus can better handle the characteristics of non-local information in the image the image denoising models of different fractional partial differential equations are proposed. Literature used the concepts of fractional calculus and differential curvature to describe the intensity of images in order to solve the problems of “staircase effect”, “speckle effect” and loss of texture details in the denoising model based on traditional integer-order partial differential equations. Variety. The fractional derivative information of the image can well handle the texture information in the image, and a good compromise is achieved between eliminating the speckle effect and suppressing the staircase effect. Furthermore, in order to effectively distinguish slopes and edges, literature constructed differential curvature along the gradient direction of the image and the second derivative perpendicular to the gradient. This model effectively eliminates the speckle effect and the staircase effect, while also better retaining the texture information such as edges in the image.

5. Conclusion

Fractional partial differential equations have obtained preliminary research results in the field of image processing, mainly focusing on the areas of image denoising and image super-resolution reconstruction. This article introduces the relationship between fractional calculus and image features: Fractional calculus can effectively retain very low frequency signals, medium frequency signals are enhanced, and high frequency signals are significantly enhanced. Furthermore, the role of fractional partial differential equations in image denoising and image super-resolution reconstruction is analyzed. Some models are mainly analyzed, and the existence, uniqueness, stability, and numerical solutions of fractional partial differential equations are discussed. Finally, The advantages and disadvantages of the current model are pointed out, and the application prospect is discussed and analyzed.

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